Qualify Exam Math 708, 709

FALL 2017 ID number

Name: ____________________

1. (20pts)

Suppose that $g : [a, b] \mapsto [a, b]$ is continuous on the real interval $[a, b]$ and is a contraction in the sense that there exists a constant $\lambda \in (0, 1)$ such that

$$|g(x) - g(y)| \leq \lambda |x - y|, \forall x, y \in [a, b] \tag{1}$$

Prove that there exists a unique fixed point in $[a, b]$ and that the fixed point iteration $x_{n+1} = g(x_n)$ converges to it for any $x_0 \in [a, b]$. And prove that the error is reduced by a factor of at least λ from each iteration to the next.

2. (20pts)

(a) Construct Polynomial p_3 that passes the points $(1, 2), (2, 10), (-1, -3)$

(b) Construct Polynomial p_3 that passes the points $(1, 0), (2, -3)$ and $p_3'(1) = 1, p_3'(2) = -1$.

(c) Construct orthogonal Polynomial of degree $0, 1, 2$ on the interval $(0, 1)$ with the weight $w(x) = -ln(x)$.

3. (20pts)

(1) Determine the nodes and weights in the 2-node Gaussian quadrature formula

$$\int_0^\infty f(x)e^{-x}dx = w_1 f(x_1) + w_2 f(x_2) \tag{2}$$

(2) Construct the polynomial of best approximation of degree 2 in the L^2 norm to the function $f(x) = \sin x$ over $[-\pi/2, \pi/2]$.

4. (20pts)

$$T_k = \begin{pmatrix} a_1 & b_2 & & & & \\ c_2 & a_2 & b_3 & & & \\ & c_3 & a_3 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_k \\ & & & & c_k & a_k \end{pmatrix}$$

The characteristic polynomial of T_k is given by $p_k(\lambda) = det(\lambda I - T_k)$

(1) Define p_k in terms of p_{k-1} and p_{k-2}.

(2) Show that if $c_j b_j > 0$ for $j = 2, \cdots, k$, then $p_k(\lambda) = 0$ only has real roots. (hint: find a real similarity transformation that symmetrizes T_k)

5. (20pts) (1) Prove: $n \times n$ matrix is normal if and only if it has n orthonormal eigenvectors.

(2) Prove that if the matrix A has orthogonal columns, then $Ax = b$ has a unique least squares solution, and find this unique solution.

6. (20pts) Use finite difference method to find the coefficient matrix for Poisson equation of 1D

$$-\frac{d^2 v(x)}{dx^2} = f(x), 0 < x < 1$$

with boundary condition $v(0) = v(1) = 0$;

and 2D Poisson equation

$$-\frac{d^2 v(x,y)}{dx^2} - \frac{d^2 v(x,y)}{dy^2} = f(x,y), 0 < x < 1, 0 < y < 1$$

with boundary condition $v(x, y) = 0$ on the boundary square.